Journal of Applied Mathematics and Mechanics 72 (2008) 48–53

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Methods of analysing ropes. The extension–torsion method☆

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ARTICLE INFO

Article history:
Received 9 December 2005

ABSTRACT

Two new approaches are used for calculating the stress–strain state of a rope and its stiffnesses. The first approach relies on the theory of fibrous composites and Saint-Venant's solution for a cylinder with helical anisotropy. The second approach is based on the solution by the finite element method of the three-dimensional problem of elasticity theory for a solid inhomogeneous cylinder formed by a finite number of elastic fibres arranged in helical lines and connected by a weak filler (in the sense that its Young's modulus is several orders of magnitude less than the Young's modulus of the fibre). The behaviour of the stiffness when the modulus of elasticity of the filler tends to zero is analysed, and the results of the limiting transition are discussed. The numerical results obtained are compared with calculations by other well-known applied theories.

There are various rope constructions that differ chiefly in the method of interweaving and in the cross-section profile of the wire from which the rope is twisted.[1] This variety is caused by the different rope service conditions. Round steel single-weave rope and double-weave rope have been most widely used. In single-weave rope, the fibres are arranged in several layers in helical lines about a central rectilinear fibre. Double-weave rope is woven from strands; an individual strand comprises a single-weave rope whose central fibre is arranged in a helical line.

Two principal approaches to constructing an elementary theory of single-weave rope are known (below, single-weave rope will be referred to simply as "rope"). One of these approaches[1,2] relies on the representation of the rope as a discrete system of curvilinear rods and uses methods of structural mechanics. The second approach is based on the equations of an elastic continuum with curvilinear anisotropy.[3,4]

With any of these approaches, the relation between the integral characteristics of the stress–strain state – the longitudinal force P_z, the torque M_z, the longitudinal strain ε and the relative twisting angle φ – has the form

$$d_{11}\varepsilon + d_{12}\varphi = P_z, \quad d_{12}\varepsilon + d_{22}\varphi = M_z \tag{0.1}$$

From this it follows that:

(a) In the general case, the longitudinal force, in addition to the longitudinal strain, generates torsion, and the torque, in addition to twisting, generates longitudinal strain;

(b) The stiffness in tension, B_r, and the stiffness in torsion, B_c, depend considerably on the way in which the ends of the rope are fastened; for example, if the ends of the rope are fastened such that $\varphi = 0$, then $B_r = d_{11}$ if $M_z = 0 - B_r = d_{11} - d_{12}^2/d_{22}$.

Unlike rectilinear rods, where all known approaches to constructing their elementary theory (the method of hypotheses, Saint Venant's theory, and asymptotic methods of elasticity theory) give the same result $B_r = d_{11} = ES$, where E is Young's modulus and S is the cross-section area, in the theory of ropes different approaches lead to different analytical expressions for the elements of the stiffness rigidity matrix d_{ij}.

To illustrate the variety of available formulae, we will present expressions for d_{11} according to the data of different authors.

By the methods of elasticity theory, Gegauff (1907)[3] obtained

$$d_{11} = ES_0\cos^2\alpha \tag{0.2}$$

Using methods of structural mechanics, Dinik (1957)[2] and Glushko (1996)[1] obtained

$$d_{11} = ES_0\cos^4\alpha \tag{0.3}$$

$$d_{11} = \sum_{i=1}^{n}(ES_i\cos^3\alpha_i + r_i^{-2}EI_i\sin^4\alpha_i\cos^3\alpha_i + r_i^{-2}GI_{pi}\sin^6\alpha_i\cos^2\alpha_i) \tag{0.4}$$

where S_0 is the effective cross-sectional area of the rope, i.e., the combined cross-section area of the individual fibres, α is the angle

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 1, pp. 81–90, 2007.

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doi:10.1016/j.jappmathmech.2008.03.012

between the tangent to the outer fibre and the axis of the rope, n is the number of fibres in the rope, G is the shear modulus, S_i is the cross-section area, I_i is the moment of inertia of the section about the axis lying in the cross-section, I_{pi} is the polar moment of inertia of the i th fibre, r_i is the distance between the axis of the rope and the fibre, and α_i is the angle of inclination of the fibre to the rope axis.

By the method of the theory of elasticity of a helically anisotropic incompressible body, Musalimov and Mokryak (1983)[4] obtained

$$d_{11} = ES_0\Phi(\alpha)$$

$$\Phi(\alpha) = 1 + \frac{3}{2}a + \frac{9}{2}b\cos^2\alpha + (9b + 3a)\,\text{ctg}^2\alpha\ln\cos\alpha,$$

$$a = 1 - 3\frac{1-\nu}{2-\nu},\quad b = \frac{1}{2-\nu} - \frac{2G}{E}$$

where ν is Poisson's ratio (see the remarks below concerning formulae (1.2)).

In the present study, two new approaches are used to determine the stress–strain state of a rope and its stiffness d_{ij}.

1. Principal relations of elasticity theory in a helical coordinate system

We will explain the principal ideas behind the first approach to constructing an elementary theory of a rope, relying on the theory of fibrous composites and Saint Venant's solution[5–7] for a cylinder with helical anisotropy. Suppose that, on a cylindrical surface of small radius, layers of elastic fibres are wound along helical lines, so that the pitch h of the helical line and the twist $\tau = 2\pi h$ remain constant. At the same time as the winding, the layers are coated with a polymer binder. After polymerization of the binding layers, we obtain a cylinder of fibrous composite. Let E_1 and ν_1 be Young's modulus and Poisson's ratio of the fibres, and let E_2 and ν_2 be Young's modulus and Poisson's ratio of the filler.

To describe the integral elastic properties of such a cylinder, we will proceed as follows.

At the geometric centre of gravity of one of the ends of the cylinder we will place the origin of a Cartesian system of coordinates x_1, x_2, x_3; we will call this the principal system of coordinates. We will introduce a helical system of coordinates r, θ, z connected to the principal system of coordinates by the relations

$$x_1 = r\cos(\theta + \tau z),\quad x_2 = r\sin(\theta + \tau z) \tag{1.1}$$

which, when r = const and θ = const, are the parametric equations of a helical line.

We will represent the radius vector of points of the helical line in the form

$$\mathbf{R} = r\mathbf{e}_1' + z\mathbf{e}_2'$$

Here

$$\mathbf{e}_1' = \mathbf{e}_r = \mathbf{i}_1\cos(\theta + \tau z) + \mathbf{i}_2\sin(\theta + \tau z),$$

$$\mathbf{e}_2' = \mathbf{e}_z = -\mathbf{i}_1\sin(\theta + \tau z) + \mathbf{i}_2\cos(\theta + \tau z)$$

where $\mathbf{i}_1$ and $\mathbf{i}_2$ are the unit vectors of the principal system of coordinates. We will connect with the helical line the basis (Frenet reference frame) $\mathbf{e}_1 = \mathbf{n}$, $\mathbf{e}_2 = \mathbf{b}$, $\mathbf{e}_3 = \mathbf{t}$, that is, the unit vectors of the principal normal, the binormal and the tangent respectively. The orthogonal matrix of the transition from the basis $\mathbf{e}_j$ to the basis $\mathbf{e}'_i$ has the form

$$A = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -g & xg \\ 0 & xg & g \end{Vmatrix};\quad g^2 = (1 + x^2)^{-1},\quad x = r\tau \tag{1.2}$$

The material of the cylinder obtained by the method described above is inhomogeneous, but, for a sufficiently large number of wound layers, by averaging theory,[8] it can at any point of the cylinder be considered as locally transversely isotropic; here, the principal axis of symmetry is directed along the vector $\mathbf{e}_3 = \mathbf{t}$.

To describe the elastic properties of the cylinder, we will use the vector matrix form of the generalized Hooke's law[9]

$$\boldsymbol{\sigma} = \mathbf{C}\mathbf{e};\quad \mathbf{e} = [e_1, \ldots, e_6]^T,\quad \boldsymbol{\sigma} = [\sigma_1, \ldots, \sigma_6]^T,\quad \mathbf{C} = (c_{ij}),\quad c_{ji} = c_{ij}$$

$$\sigma_k = \sigma_{kk},\quad k = 1, 2, 3,\quad \sigma_4 = \sigma_{23},\quad \sigma_5 = \sigma_{13},\quad \sigma_6 = \sigma_{12}$$

$$e_k = e_{kk},\quad k = 1, 2, 3,\quad e_4 = 2e_{23},\quad e_5 = 2e_{13},\quad e_6 = 2e_{12}$$

where σ_{ij} and e_{ij} are the tensor components of the stresses and small strain components respectively.

It is well known[9] that the elastic properties of a transversally isotropic material are defined by five technical constants: Young's moduli E and E', Poisson's ratios ν and ν' and the shear modulus G'. The elements of the matrix $\mathbf{C}$ are expressed in terms of these constants by the formulae

$$c_{11} = c_{22} = \frac{E(E' - E\nu'^2)}{\gamma(1+\nu)},\quad c_{12} = \frac{E(E'\nu + E\nu'^2)}{\gamma(1+\nu)}$$

$$c_{13} = c_{23} = \frac{EE'\nu'}{\gamma},\quad c_{33} = \frac{E'^2(1-\nu)}{\gamma},$$

$$c_{66} = \frac{E}{2(1+\nu)},\quad \gamma = E'(1-\nu) - 2E\nu'^2$$

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = 0,\quad c_{44} = c_{55} = G' \tag{1.2}$$

By averaging theory,[8] we have

$$E = \left[k_1\frac{1-\nu_1^2}{E_1} + k_2\frac{1-\nu_2^2}{E_2} + \frac{\nu'^2}{E'}\right]^{-1},$$

$$\nu = E\left[k_1\frac{\nu_1 - \nu_1^2}{E_1} + k_2\frac{\nu_2 - \nu_2^2}{E_2} + \frac{\nu'^2}{E'}\right]$$

$$G' = \left[2k_1\frac{1+\nu_1}{E_1} + 2k_2\frac{1+\nu_2}{E_2}\right]^{-1};\quad E' = k_1E_1 + k_2E_2,$$

$$\nu' = k_1\nu_1 + k_2\nu_2,\quad k_1 + k_2 = 1 \tag{1.3}$$

where k_1 and k_2 are the concentrations over the section perpendicular to the unit vector $\mathbf{e}_3$ of the bearing elements and filler respectively.

As a result of the transition from the basis $\mathbf{e}_j$ to the basis $\mathbf{e}'_i$, we obtain the following relations of the generalized Hooke's law in a helical system of coordinates

$$\sigma_{rr} = \Sigma_1,\quad \sigma_{\theta\theta} = \Sigma_2,\quad \sigma_{zz} = \Sigma_3,\quad \sigma_{\theta z} = \Sigma_4$$

$$\Sigma_l = c'_{l1}e_{rr} + c'_{l2}e_{\theta\theta} + c'_{l3}e_{zz} + 2c'_{l1}e_{\theta z},\quad l = 1, 2, 3, 4$$

$$\sigma_{rz} = 2c'_{55}e_{rz} + 2c'_{56}e_{r\theta},\quad \sigma_{r\theta} = 2c'_{56}e_{rz} + 2c'_{66}e_{r\theta} \tag{1.4}$$

$$c'_{11} = c_{11},\quad c'_{12} = (c_{12}+c_{13}x^2)g^2,\quad c'_{13} = (c_{13}+c_{12}x^2)g^2,$$
$$c'_{14} = (c_{13}-c_{12})xg^2$$
$$c'_{22} = [c_{11}+2(c_{13}+c_{44})x^2+c_{33}x^4]g^4$$
$$c'_{23} = [c_{13}+(c_{11}+c_{33}-4c_{44})x^2+c_{13}x^4]g^4$$
$$c'_{24} = [(c_{13}-c_{11}+2c_{44})x+(c_{33}-c_{13}-2c_{44})x^3]g^4$$
$$c'_{33} = [c_{33}+2(c_{13}+2c_{44})x^2+c_{11}x^4]g^4$$
$$c'_{34} = [(c_{33}-c_{13}-2c_{44})x+(2c_{13}-2c_{11}+c_{44})x^3]g^4$$
$$c'_{44} = [c_{44}-(2c_{13}-c_{11}-c_{33}+2c_{44})x^2+c_{44}x^4]g^4$$
$$c'_{55} = (c_{44}+c_{66}x^2)g^2,\ c'_{56} = (c_{44}-c_{66})xg^2,\ c'_{66} = (c_{66}+c_{44}x^2)g^2,$$
$$c_{66} = 2(c_{11}-c_{12}) \tag{1.5}$$

The strain tensor components in the basis of the helical system of coordinates are expressed in terms of displacements by the following relations

$$e_{rr} = \frac{\partial u_r}{\partial r},\quad e_{\theta\theta} = \frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\quad e_{zz} = Du_z$$
$$2e_{r\theta} = \frac{\partial u_\theta}{\partial r}+\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right),\quad 2e_{z\theta} = \frac{1}{r}\frac{\partial u_z}{\partial\theta}+Du_\theta,\quad 2e_{rz} = \frac{\partial u_z}{\partial r}+Du_r$$
$$D = \frac{\partial}{\partial z}-\tau\frac{\partial}{\partial\theta} \tag{1.6}$$

The equations of equilibrium in the stresses in this case have the form

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+D\sigma_{rz} = 0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+2\frac{\sigma_{r\theta}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+D\sigma_{\theta z} = 0$$
$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+D\sigma_{zz} = 0 \tag{1.7}$$

Using relations (1.4)–(1.7), we can to obtain a system of differential equations in the components of the displacement vector $\mathbf{u}=[u_r, u_\theta, u_z]^T$ that, together with the conditions of the stress-free side surface, we will write in the form

$$L(\partial_z)\mathbf{u} = 0,\quad N(\partial_z)\mathbf{u} = 0;\quad \partial_z = \frac{\partial}{\partial z} \tag{1.8}$$

Seeking the solution in the form $\mathbf{u}=\mathbf{a}(r,\theta)\exp(i\lambda z)$, after substitution into relations (1.8), we obtain the eigenvalue problem on the cross-section

$$L(i\lambda)\mathbf{a} = 0,\quad N(i\lambda)\mathbf{a} = 0 \tag{1.9}$$

with an infinite set of eigenvalues (EVs), symmetrically positioned in the complex plane λ.

Saint Venant's solution is defined by three fourfold EVs[5–7] $\lambda_0=0$, $\lambda_1^\pm=\pm\tau$; the remaining EVs are complex. Usually, the complex EVs are simple, which enables the general solution for a cylinder to be represented in the form[7,10]

$$\mathbf{u} = \sum_{n=1}^{12} C_n\mathbf{u}_n+\sum_k[C_k^+\mathbf{u}(z,\lambda_k^+)+C_k^-\mathbf{u}(z-l,\lambda_k^-)];$$
$$\mathbf{u}(z,\lambda_k) = \mathbf{a}_k\exp(i\lambda_k z) \tag{1.10}$$

where l is the length of the cylinder, C_n and $C_k^\pm$ are arbitrary constants, determined by satisfying the boundary conditions on the ends of the cylinder $z=0$, l, $\mathbf{u}_n$ are the elementary Saint Venant's solutions,[5–7] $\lambda_k^+(\mathrm{Im}\,\lambda_k^+>0)$ and $\lambda_k^-(\mathrm{Im}\,\lambda_k^-<0)$ are the EVs of the problem (1.9), and $\mathbf{a}_k^+$ and $\mathbf{a}_k^-$ are the eigenvectors corresponding to them.

Half of the elementary Saint Venant's solutions define the displacement of the cylinder as a rigid body, and the other half define the stress–strain state (SSS), equivalent in each cross-section to the longitudinal force and torque ($\lambda_0=0$), the bending moments and the transverse forces ($\lambda_1^\mp=\pm\tau$). The SSS corresponding to the second sum in expression (1.10) is self-balanced in each cross-section of the cylinder and attenuates with distance from the ends. However, with a weak filler, which is equivalent to satisfying of the condition $m=E_2/E_1 \ll 1$ (as shown by formulae (1.3), from this condition it follows that $E \ll E'$, i.e., the material with averaged characteristics possesses strong anisotropy), there are a finite number of EVs $\lambda_k^\pm$, the imaginary part of which tends to zero as $m\to 0$ (we will return to this question below when discussing the results of the calculations). The elementary solutions corresponding to such eigenvalues with the corresponding boundary conditions on the ends can have a considerable effect on the internal SSS of the cylinder and its stiffness.

The presence in the solution of the three-dimensional problem (1.10) of weakly attenuating elementary solutions leads to violation of Saint Venant's principle for cylindrical bodies of composite materials with a weak filler. Effects of this kind have been investigated mathematically in detail for laminated plates and cylinders with alternating rigid and soft layers.[11,12] A similar situation occurs in the three-dimensional solution for a rope, which below will be considered as the limiting case of the solution described, the elastic characteristics of which $E_2=0$ and $\nu_2=0$.

2. Construction of the stiffness matrix of a rope using saint venant's solution

Saint-Venant's solutions of the extension–torsion problem are described by the relations[5–7]

$$u_r = C_1a_1+C_2a_2,\quad u_\theta = C_2zr,\quad u_z = C_1z \tag{2.1}$$

$$\sigma_{rr} = S_1,\quad \sigma_{\theta\theta} = S_2,\quad \sigma_{zz} = S_3,\quad \sigma_{\theta z} = S_4$$
$$S_1 = C_1(D_la_1+c'_{l3})+C_2(D_la_2+c'_{l4}),$$
$$D_l = c'_{1l}\partial_r+r^{-1}c'_{1l};\quad l = 1,\dots,4 \tag{2.2}$$

where ∂_r is the ordinary derivative with respect to r, and $a_j=a_j(r)$ are the solutions of the following boundary-value problems

$$Za_j = F_j,\quad a_j(0) = 0,\quad D_1a_j|_{r=a} = f_j$$
$$Z = \partial_rD_1+r^{-1}(c'_{11}-c'_{12})\partial_r+r^{-2}(c'_{12}-c'_{22})$$
$$F_1 = -\partial_rc'_{13}-r^{-1}(c'_{13}-c'_{23}),\quad f_1 = -c'_{13}(a)$$
$$F_2 = -\partial_r(rc'_{14})-c'_{14}+c'_{24},\quad f_2 = -ac'_{14}(a) \tag{2.3}$$

The constants $C_1=\varepsilon$ and $C_2=\varphi$ are determined from the integral equilibrium conditions

$$2\pi\int_0^a\sigma_{zz}rdr = d_{11}C_1+d_{12}C_2 = P_z,$$
$$2\pi\int_0^a\sigma_{z\theta}r^2dr = d_{12}C_1+d_{22}C_2 = M_z \tag{2.4}$$

where P_z and M_z are the projections of the principal vector and the principal moment of the stresses acting in the cross-section onto the axis of the cylinder.

As stated above, the rope will be considered as a cylinder of composite material described above, the elastic characteristics of which $E_2=0$ and $\nu_2=0$. In this case, from relations (1.2) and (1.3) it follows that only one element of the matrix of moduli **C** will be non-zero: $c_{33}=E'=k_1E_1$. Furthermore, $\nu'=k_1\nu_1$.

Remark 1. Besides the assumption on the incompressibility of the material, in an earlier study[4] it was assumed that $G\neq 0$, while from the third relation of (1.3) it follows that $G\to 0$ when $E_2\to 0$.

We will introduce the dimensionless parameter $\eta^2=c_{11}/E'$ and analyse relations (1.2) and (1.5) as $\eta\to 0$. We have

$$c'_{11}=E'\eta^2,\quad c'_{22}=E'x^4g^4+O(\eta^2),\quad c'_{23}=E'x^2g^4+O(\eta^2),$$
$$c'_{24}=E'x^4g^4+O(\eta^2)$$
$$c'_{33}=E'g^4+O(\eta^2),\quad c'_{34}=E'xg^4+O(\eta^2),\quad c'_{44}=E'xg^4+O(\eta^2)$$
$$c'_{12}\approx c'_{13}\approx c'_{14}\approx c'_{66}=O(\eta^2) \tag{2.5}$$

Note that $c'_{13}/c'_{11}\to\nu'$ when $\eta\to 0$.

We will transform formulae (2.2) taking relations (2.5) into account, assuming that $C_2=0$ and the strains $e_{rr,j}=\partial_r a_j$ and $e_{\theta\theta,j}=a_j/r$ are of the order of unity for any $r\in[0,a]$, $\tau_0=a\tau\in[0,\infty)$ (below, this assumption will be justified). Under this assumption, with an error $O(\eta^2)$, we have

$$\sigma_{zz}=E'[C_1(x^2e_{\theta\theta,1}+1)+C_2(x^2e_{\theta\theta,2}+rx)]g^4$$
$$\sigma_{\theta z}=E'[C_1(x^3e_{\theta\theta,1}+x)+C_2(x^3e_{\theta\theta,2}+rx^2)]g^4 \tag{2.6}$$

To assess $\sigma_{\theta\theta j}$, we will consider boundary-value problems (2.3). We will first consider the case when $j=1$. We will introduce the dimensionless coordinate $\rho=r/a$ and transform the equations and boundary conditions taking relations (2.5) into account. We obtain

$$Za_1=\eta^2(a''_1+a'_1/\rho-1/\rho^2)+qa_1=F_1 \tag{2.7}$$

$$a_1(0)=0,\quad a'_1(1)=-\nu'/(1+\tau_0^2) \tag{2.8}$$

Here

$$F_1=\rho\tau_0^2\frac{1+\eta^2(1-\nu+\nu\rho^2\tau_0^2)}{(1+\rho^2\tau_0^2)^2},\quad q=-\tau_0^2\frac{\rho^2\tau_0^2+\eta^2(2+\rho^2\tau_0^2)}{(1+\rho^2\tau_0^2)^2} \tag{2.9}$$

When $\tau_0=0$, the solution of this problem has the form $a_1=-\nu'a\rho$.

For any finite τ_0 and $\eta\to 0$, we obtain a singular problem: the case of a differential equation with a small parameter for the highest derivative.

To construct the solution in this case, we will use the asymptotic method. The solution will be sought in the form

$$a_1=a_1^0+a_1^1;\quad a_1^0=F_1/q=-1/(\rho\tau_0^2) \tag{2.10}$$

where a_1^0 is the degenerate solution, which is obtained on the basis of the inhomogeneous equation (2.7) if we part $\eta=0$ in it, and a_1^1 is the correcting solution, which is constructed such that solution (2.10) satisfies boundary conditions (2.7).

The quantity a_1^1 is constructed as follows.

We transform the homogeneous equation (2.7) using replacement of the variable, assuming $\rho=\eta\xi$. We have

$$Z_0a_1^1=\frac{d^2a_1^1}{d\xi^2}+\frac{1}{\xi}\frac{da_1^1}{d\xi}-\frac{a_1^1}{\xi^2}+\eta^2Z_1a_1^1=0 \tag{2.11}$$

where Z_1 is the linear operator restricted for all τ_0 and ρ and having the order of unity with respect to the parameter η. The structure of Eq. (2.11) enables as when constructing its solution, to apply the small-parameter method, formally seeking the solution in the form of a power series in the parameter η. In the zero the approximation we obtain

$$a_1^1=A_1\xi+A_2\xi^{-1} \tag{2.12}$$

where A_1 and A_2 are arbitrary constants. If we put

$$A_1=-a\nu\eta/(1+\tau_0^2),\quad A_2=1/(\eta\tau_0^2)$$

then, after changing to the variable ρ, on the basis of relations (2.10) and (2.12) we obtain the solution

$$a_1=-\nu a\rho/(1+\tau_0^2) \tag{2.13}$$

which satisfies both boundary conditions (2.8). By a similar method we obtain

$$a_2=-\nu\tau_0a^2\rho^3/(1+\tau_0^2) \tag{2.14}$$

Correspondingly, we have

$$e_{\theta\theta,1}=-\nu/(1+\tau_0^2),\quad e_{\theta\theta,2}=-\nu\tau_0a\rho^2/(1+\tau_0^2) \tag{2.15}$$

The formulae for the stiffnesses d_{ij} are obtained by substituting expressions (2.15) into (2.6), and then into conditions (2.4). After evaluating of the integrals, assuming $\tau_0=\operatorname{tg}\alpha$, we arrive at the expressions

$$d_{1j}=\pi k_1a^{1+j}E_1D_{1j},\ d_{22}=\pi k_1a^4E_1D_{22},$$
$$D_{1j}=i_{j-1}-si_j,\ j=1,2,\ D_{22}=i_2-si_3 \tag{2.16}$$

$$i_0=\cos^2\alpha,\quad i_1=-\cos^2\alpha\operatorname{tg}^{-4}\alpha(\operatorname{tg}^2\alpha+\cos^2\alpha+\ln\cos^2\alpha)$$
$$i_2=\cos^2\alpha\operatorname{tg}^{-6}\alpha(2\operatorname{tg}^2\alpha+\operatorname{tg}^4\alpha-2\cos^2\alpha\ln\cos^2\alpha)$$

$$i_3=-\cos^2\alpha\operatorname{tg}^{-8}\alpha(6\operatorname{tg}^2\alpha+3\operatorname{tg}^4\alpha-\operatorname{tg}^6\alpha+6\cos^2\alpha\ln\cos^2\alpha)$$

$$s=\nu\sin^2\alpha$$

Remark 2. For steel rope, $\alpha=10$–$18°$. With such values of α, sufficient accuracy is ensured by the following approximate formulae, stemming from formulae (2.16)

$$d_{11}=\pi k_1a^2E_1[1-(1+\nu/2)\sin^2\alpha]$$

$$d_{12}=\pi k_1a^3E_1\operatorname{tg}\alpha[1-(4/3+\nu)\sin^2\alpha]/2$$

$$d_{22}=\pi k_1a^4E_1\operatorname{tg}^2\alpha[1-(3/2+3\nu/4)\sin^2\alpha]/3$$

Remark 3. In some rope constructions, to reduce abrasive wear by dust getting into the space between fibres, wires with a non-circular cross-section profile are used. In formulae (2.16), this factor is taken into account by the parameter k_1, which is equal to the ratio of the combined area of cross-sections of the fibres to the cross-section area of the rope as a circular cylinder. For wires of circular cross-section, $k_1=\pi/4$.

To conclude this section, we will give formulae for calculating the stresses in the case when $C_2=0$ (no twisting of the rope). We have

$$\sigma_{\theta\theta}=p\rho^2\sin^2\alpha\cos^2\alpha,\quad \sigma_{zz}=p\cos^4\alpha,\quad \sigma_{\theta z}=p\rho\sin\alpha\cos^3\alpha$$
$$p=(1-\nu\rho^2\sin^2\alpha)P_z/(\pi a^2D_{11}) \tag{2.17}$$

3. Some certain results of a numerical analysis

In the calculations, two objectives were pursued: to give a comparative analysis of the values of the stiffnesses of the rope, calculated by means of formulae (0.2) to (0.4) and (2.16), and to compare these results with the results attained from the solution of the three-dimensional problem of elasticity theory for an inhomogeneous cylinder of a composite reinforced with helical rigid spirals with a weak filler. Calculations of the three-dimensional SSS and stiffnesses were carried out by the finite element method (FEM). Over the course of the investigation, a numerical analysis was made of the behaviour of the stiffnesses of a solid cylinder when Young's modulus of the filler, E_2, tends to zero. In parallel, a similar analysis was conducted by numerical integration of problems (2.3) with subsequent determination of the stiffnesses d_{ij}.

A two-layer rope with the following parameters was chosen for the calculation:

number of fibres $N = 1 + 6 + 12 = 19$;
diameter of the rod $d_p = 0.0011$ m (in calculations with a filler, it was this value that was adopted, to avoid bridging of a filler with zero thickness);
pitch of the helical line $h = 0.072$ m;
Young's modulus of the bearing elements $E_1 = 2 \times 10^{11}$ Pa;
Poisson's ratios of the bearing elements and filler $\nu_1 = \nu_2 = 0.3$;
$2a = 0.0059$ m (note that $2a \neq 5d_p$);
$k_1 = 19d_p^2/(4a^2) = 0.66$, $\alpha = 14.44^\circ = 0.2519$ (rad).

As a result of calculations by means of formulae (2.16), the following values were obtained:

$d_{11} = 3.353 \times 10^6$ Pa, $d_{12} = 1257$ N/m, $d_{22} = 0.6293$ N

For comparison, we will give the results of a calculation by means of (0.2)–(0.4), and also the results for d_{12} and d_{22} (Ref. 1):

$d_{11} = 3.384 \times 10^6$ Pa (Ref. 3), $d_{11} = 3.174 \times 10^6$ Pa (Ref. 2)
$d_{11} = 3.475 \times 10^6$ Pa, $d_{12} = 1092$ N/m, $d_{22} = 0.7348$ N (Ref. 1)

In accordance with the above, two problems were solved.

Problem 1. Using the finite elements method, the three-dimensional problem was solved for a solid cylinder of a composite (without using the averaging method) with the above values of the parameters for different values of the ratio $m = E_1/E_2$, with the following boundary conditions on the ends

$$z = 0:\ u_r = u_\theta = u_z = 0; \quad z = h:\ u_r = u_\theta = 0, \quad u_z = 0.005 \text{ м} \tag{3.1}$$

The SSS and the rigidities d_{ij} were determined.

Problem 2. By numerical integration of problem (2.7), (2.8), the SSS of Saint-Venant's problem and the stiffnesses d_{ij} were calculated.

The results obtained for d_{11} are given below:

m	10^{-2}	10^{-3}	10^{-5}	10^{-7}	0
Problem 1					
d_{11}, 10^6 Pa	3.349	3.175	1.107	0.341	0.308
Problem 2					
d_{11}, 10^6 Pa	3.326	3.306	0.774	–	–

Remark 4. When solving Problem 1 in the limiting case $m = 0$, due to of the presence of gaps, there were no contact interactions between the fibres, while in Saint-Venant's solution there is interaction, as, according to the first formula of (2.17), $\sigma_{\theta\theta} \neq 0$. The absence of contact between the helical fibres leads, as shown by calculations, to a considerable reduction in d_{11}. To take into account the contact interactions between the fibres within the framework of the finite-element method, however, it is necessary to solve an unjustifiably complex problem, although the formulation of such a problem is not itself ruled out and deserves separate attention.

Remark 5. When $m < 10^{-4}$, in the numerical integration of problems (2.7), (2.8), stability was lost on account of the smallness of the parameter $\eta^2 = O(m)$.

From calculations based on the finite-element method with $m = 0$ it follows that the strain is periodic in nature, with a period equal to the pitch of the helical spirals. In Saint Venant's solution, the pattern of the strain state remains constant in all cross-sections and the rope retains its cylindrical shape.

Certain considerations concerning this qualitative discrepancy in the strain pattern of the rope can be stated.

In constructing the finite-element method solution, the boundary-value problem with rigid fixing at both ends of the rope was solved, one end remaining stationary and the second being moved forwards along its axis. However, Saint Venant's solution with $z = 0$ satisfies only one of the three conditions in (3.1), namely $u_z = 0$, and the radial displacement $u_r = C_1 a_1(r) + C_2 a_2(r) \neq 0$, where the $a_j(r)$ are determined by the solutions of boundary-value problems (2.3). For normal cylindrical rods, this difference in boundary conditions when determining the SSS far from the ends is of no fundamental importance, as the SSSs arising at the ends are statically equivalent by virtue of Saint Venant's principle. For a rope, as noted above, in the case of Problem 1, Saint Venant's principle is not satisfied by virtue of the absence of internal bonds between the individual fibres. The absence of such bonds for a helical fibre not coinciding with the axial fibre of the rope, with forward movement of its ends parallel to the axis of the rope, leads to the emergence, apart from the tensile force, of a bending moment and a shearing force. It can be shown that the strain corresponding to the bending moment and shearing force is proportional to cos τz and sin τz ($\tau = 2\pi/h$) respectively, which corresponds to the pattern obtained using the finite-element method. Here, the overall bending moment and shearing force in the cros-section of the rope are zero.

Thus, calculations of the stiffness d_{11} for a steel rope by means of formula (0.4), obtained by methods of structural mechanics, and by means of formula (2.16), obtained by methods of the theory of composites using Saint Venant's solution, give similar results, differing in the 3% range. For the remaining stiffnesses, the difference in the calculations reaches 20%.

Numerical analysis of the problem using the finite-element method showed that the stiffness of the rope depends considerably on the contact interactions between the individual fibres. From this it follows that any variation in contact interaction over the length (for example, the back of contact between the individual fibres on certain sections as a result of assembly or operating conditions) can cause longitudinal inhomogeneity of its structure, which, in turn, leads to a concentration of stresses due to bending of individual fibres and can cause failure. A possible reason for delamination of the rope is the "untwisting" effect, the magnitude of which, besides technological factors, depends on the value of the stiffness d_{12}. To reduce this effect, double-weave rope is used.

A general analysis indicates that the SSS of a rope depends considerably on the method used to fasten its ends.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (04-01-00069, 07-01-00254a) and the Southern Federal University.

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Translated by P.S.C.